On the Order Sampling Design

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1 Introduction

In the current paper we study order sampling introduced by Roseń (1997a). The sampling procedure is based on the order of generated random variables. The design allows unequal inclusion probabilities and gives fixed sample size.

The uniform order sampling is used for Swedish Consumer Price Index (Ohlsson 1998). Although easy to apply, the exact probability law of the order sampling design was until recently unknown. The estimators under this design were built on approximate inclusion probabilities called target inclusion probabilities.

Traat, Bondesson and Meister (2000) derived the exact probability function of the order sampling design which made it possible to calculate the exact inclusion probabilities for different order sampling designs. In bachelor thesis (Rajaleid 2000) computational aspects of order sampling design were considered. Earlier, Aires (1998) has given recursive formulae for inclusion probabilities of Pareto order sampling design.

In this paper the probability function of the order sampling design is derived with an alternative method. Two different forms of the final result are presented. A deeper attention is paid to the uniform order sampling and the behaviour of the estimators and their variance estimators under that design. We employ the exact inclusion probabilities which now can be calculated directly from the formula of sampling design by standard methods. The software Mathcad is used for calculations (the output is given in Appendices).

In this paper one more useful application of order sampling design is presented. Recently Kröger, Särndal and Teikari (1999) have proposed a new sampling design, Pomix-sampling, which has inclusion probabilities in a form of a mixture of equal and unequal inclusion probabilities. A shortcoming of their design is variable sample size. We use mixture-type inclusion probabilities in our order sampling design and overcome the shortcoming of Pomix-sampling. We study the effect of the mixing parameter to the estimator variance in an example. We notice the same surprising phenomena that the variance is not minimized with inclusion probabilities strictly proportional to size.

2 Basic concepts

Let $U = \{1, 2, ..., N\}$ be a finite population and $\boldsymbol{I} = (I_1, I_2, ..., I_N)$ a random design vector on U describing sampling without replacement. Then

$$I_i = \begin{cases} 1, & \text{if i is sampled,} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

A realisation of I is denoted by $\mathbf{k} = (k_1, k_2, ..., k_N)$, $k_i \in \{0, 1\}$. Each $\mathbf{k}$ describes a randomly selected sample.

The multivariate distribution of $\mathbf{I}$, $p(\mathbf{k}) = P(\mathbf{I} = \mathbf{k})$, is called sampling design (Traat 2000) which in our case is a multivariate Bernoulli distribution.

The first and second order inclusion probabilities of population elements can be computed as follows:

$$\pi_i = \sum_{\mathbf{k}:k_i=1} p(\mathbf{k}), \tag{2}$$

$$\pi_{ij} = \sum_{\mathbf{k}:k_i=k_j=1} p(\mathbf{k}), \tag{3}$$

where the summation is taken over all the points $\mathbf{k}$ in which ith (or ith and jth) coordinate are kept fixed to 1.

Let $\mathbf{y}$ be a study variable on U. The unbiased estimator of population total $Y = \sum y_i$ is

$$\hat{Y}_\pi = \sum_U \frac{1}{\pi_i} y_i I_i. \tag{4}$$

In this paper we consider fixed size n sampling designs. Then the variance of $\hat{Y}_\pi$ is (Särndal, Swensson and Wretman 1992)

$$Var\hat{Y}_\pi = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Big(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j} \Big)^2 \Big(\pi_i \pi_j - \pi_{ij} \Big), \tag{5}$$

and the unbiased estimator of the variance is

$$\hat{Var}\hat{Y}_\pi = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Big(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j} \Big)^2 \Big(\frac{\pi_i \pi_j}{\pi_{ij}} - 1 \Big) I_i I_j. \tag{6}$$

3 Sampling with unequal inclusion probabilities

3.1 Inclusion probabilities proportional to size

It is useful to draw a sample from the population with probabilities proportional to the variable $\mathbf{y}$. In this case the π-estimator (4) would be precise for any sample and the variance of the estimator would be zero.
The needed probabilities would be

$$\pi_i = \frac{ny_i}{\sum_U y_j}. \tag{7}$$

In a real survey the values of $\mathbf{y}$ are unknown. If the values of another variable (size variable) $\mathbf{x}$ are known for each element i and it is believed that x_i is approximately proportional to y_i, one can compute the probabilities, using the values x_i:

$$\pi_i = \frac{nx_i}{\sum_U x_j}. \tag{8}$$

It may be true for some elements that $\pi_i > 1$. Then let $\pi_i = 1$ for all such elements and the rest of the inclusion probabilities are

$$\pi_i = (n - n_A)\frac{x_i}{\sum_{U \setminus A} x_j}, \tag{9}$$

where A is the set of n_A elements such that $nx_i > \sum_U x_j$.

3.2 Mixed inclusion probabilities

Kröger, Särndal and Teikari (1999) have introduced a new method called Poisson Mixture (Pomix) sampling. It is a Poisson sampling with mixed inclusion probabilities, so that for the extremes of the mixing parameter it becomes Poisson πps sampling or Bernoulli sampling.

Inclusion probabilities for Pomix sampling are calculated as follows:

$$\lambda_i^\mu = \mu\lambda_0 + (1 - \mu)\lambda_i, \tag{10}$$

where $\lambda_0 = \frac{n}{N}$ is the Bernoulli inclusion probability, λ_i is the πps inclusion probability calculated by (8)-(9), and $\mu \in [0, 1]$ is the mixing parameter.

As seen from (10) $\lambda_i^\mu = \lambda_0$ for all i if $\mu = 1$, and $\lambda_i^\mu = \lambda_i$ if $\mu = 0$.

As stated in Kröger, Särndal and Teikari (1999), Pomix sampling is easy to use for regulating response burden with permanent random numbers. Monte Carlo study showed that the variance of $\hat{Y}$ does not attain its minimum with $\mu = 0$, but with μ having value between 0.02 and 0.03. A shortcoming of Pomix-sampling is the variable sample size.

We calculate the inclusion probabilities as described for Pomix sampling, but use them in order sampling scheme. The result is a fixed size design which in one end of the mixing parameter is a simple random sampling and in another end a probability-proportional-to-size sampling.

3.3 Order sampling scheme

A convivient possibility to draw a sample with fixed size and unequal inclusion probabilities is order sampling. In Rosén(1997a) the definitions are given as follows:

Definition 1. To each unit i in U a probability function F_i is associated. Independent random variables Q_i with distributions F_i are realized. The units with n smallest Q-values constitute the sample. This sampling scheme is called *order sampling* and denoted by $OS(n; \mathbf{F})$.

Definition 2. Let $H(t)$ be a probability function and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ the target inclusion probabilities. Then the $OS(n; \mathbf{F})$-scheme with $F_i(t) = H(t \cdot H^{-1}(\lambda_i))$ is called *order sampling with probability proportional to size* and denoted by $OS\pi ps(n, H, \lambda)$.

The target inclusion probabilities λ_i are different from the exact inclusion probabilities π_i but converge asymptotically to π_i (Roseń 1998). Depending on the shape distribution three special order sampling schemes are considered in the literature.

Uniform order sampling (earlier called sequential Poisson sampling by Ohlsson):

$$H(t) = \begin{cases} t, \text{ if } 0 \leq t \leq 1, \\ 1, \text{ if } t \geq 1; \end{cases} \tag{11}$$

$$h(t) = \begin{cases} 1, \text{ if } 0 \leq t \leq 1, \\ 0, \text{ if } t \geq 1; \end{cases} \tag{12}$$

$$H^{-1}(\lambda_i) = \lambda_i, i = 1, 2, ..., N. \tag{13}$$

Exponential order sampling (earlier called successive sampling by Hájek):

$$H(t) = 1 - e^{-t}, \; h(t) = e^{-t}, 0 \leq t \leq \infty, \tag{14}$$

$$H^{-1}(\lambda_i) = ln(1-\lambda_i), i = 1, 2, ..., N. \tag{15}$$

Pareto order sampling:

$$H(t) = \frac{t}{1+t},\ h(t) = \frac{t}{(1+t)^2}, 0 \le t \le \infty, \tag{16}$$

$$H^{-1}(\lambda_i) = \frac{\lambda_i}{1-\lambda_i}, i = 1, 2, ..., N. \tag{17}$$

It is shown in Rosén (1997b) that the variance of the estimator of the population total is minimized by Pareto shape distibution. Inclusion probabilities for Pareto order sampling have been given by recursive formulae in Aires (1998).

We concentrate on the uniform case because generating random numbers from the uniform distribution is most frequently used in practice for drawing a sample. At the same time theoretical properties of uniform order sampling design have not been studied much.

4 Order sampling design

4.1 General expression

Although easy to apply, the exact expression of the order sampling design was until recently unknown. The general expression for the probability function of the order sampling design was derived in Traat, Bondesson and Meister (2000). Here we present an alternative derivation and two alternative forms of the final result.

Consider population $U = \{1, 2, \cdots, N\}$ and order distributions $\boldsymbol{F} = (F_1, F_2, \ldots, F_N)$ for the elements of U. We are looking for sampling design

$$p(\boldsymbol{k}) = P(\boldsymbol{I} = \boldsymbol{k}). \tag{18}$$

To draw a sample, ranking variables $Q_1, Q_2, \ldots, Q_N$ from distributions $F_1, F_2, \ldots, F_N$ are realized. Population elements with n smallest Q-values constitute the sample.

Let us consider a realization $\boldsymbol{k} = (k_1, k_2, \ldots, k_N)$. By $\boldsymbol{k}$ the population is devided into two subsets G_1 ja G_2:

$$G_1 = \{i : k_i = 1\},$$

$$G_2 = \{j : k_j = 0\}.$$

Subset G_1 consists of n sampled elements and G_2 contains the rest $N-n$ elements. The biggest Q-value for the elements of subset G_1 is bigger than the smallest Q-value for the elements of subset G_2.

Thus order sampling design is

$$p(\boldsymbol{k}) = P(\max_{i \in G_1} Q_i < \min_{j \in G_2} Q_j) = \int_0^\infty F_{max}(t) f_{min}(t) dt, \tag{19}$$

where $F_{max}(t)$ is the distribution of $\max_{i \in G_1} Q_i$ and $f_{min}(t)$ is the density of $\min_{j \in G_2} Q_j$. The distribution and density functions of $\max_{i \in G_1} Q_i$ and $\min_{j \in G_2} Q_j$ are:

$$F_{max}(t) = \prod_{i=1}^{n} F_i(t), \tag{20}$$

$$F_{min}(t) = 1 - \prod_{i=1}^{n} \{1 - F_i(t)\}, \tag{21}$$

$$f_{max}(t) = \sum_{i=1}^{n} f_i(t) \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(t), \tag{22}$$

$$f_{min}(t) = \sum_{i=1}^{n} f_i(t) \prod_{\substack{j=1 \\ j \neq i}} [1 - F_j(t)]. \tag{23}$$

Substituting these expressions into (19) we get:

$$p(\boldsymbol{k}) = \int_0^\infty \prod_{i \in G_1} F_i(t) \{ \sum_{j \in G_2} f_j(t) \prod_{\substack{l \in G_2 \\ l \neq j}} [1 - F_l(t)] \} dt. \tag{24}$$

Further, we may write

$$p(\boldsymbol{k}) = \sum_{j \in G_2} \int_0^\infty f_j(t) \prod_{\substack{l \in G_2 \\ l \neq j}} [1 - F_l(t)] \prod_{i \in G_1} F_i(t) dt. \tag{25}$$

Formula (25) gives us the probability of $\boldsymbol{k} = (k_1, k_2, \ldots, k_N)$. Noting that

$$\prod_{i \in G_1} F_i(t) = \prod_{i=1}^{N} [F_i(t)]^{k_i},$$

$$\prod_{\substack{l\in G_2\\ l\neq j}}[1-F_l(t)] = \prod_{\substack{l=1\\ l\neq j}}^{N}[1-F_l(t)]^{1-k_l}$$

and

$$\sum_{j\in G_2}(\cdot) = \sum_{j=1}^{N}(1-k_j)(\cdot) \tag{26}$$

we get

$$p(\boldsymbol{k}) = \sum_{j=1}^{N}(1-k_j)\int_0^\infty \prod_{i=1}^{N}[F_i(t)]^{k_i}[1-F_i(t)]^{1-k_i}\frac{f_j(t)}{1-F_j(t)}dt, \tag{27}$$

Instead of (19) one may write:

$$p(\boldsymbol{k}) = P(\min_{j\in G_1} Q_j > \max_{i\in G_1} Q_i) = \int_0^\infty [1-F_{min}(t)]f_{max}(t)dt, \tag{28}$$

where $F_{min}(t)$ is the distribution of $\min_{j\in G_2} Q_j$ and $f_{max}(t)$ is the density of $\max_{i\in G_1} Q_i$. With (22) and (21) we get

$$p(\boldsymbol{k}) = \int_0^\infty \prod_{j\in G_2}[1-F_j(t)]\{\sum_{i\in G_1} f_i(t)\prod_{\substack{l\in G_1\\ l\neq i}} F_l(t)\}dt \tag{29}$$

and further

$$p(\boldsymbol{k}) = \sum_{i\in G_1}\int_0^\infty f_i(t)\prod_{\substack{l\in G_1\\ l\neq i}} F_l(t)\prod_{j\in G_2}[1-F_j(t)]dt. \tag{30}$$

Like in (27) we may write

$$p(\boldsymbol{k}) = \sum_{j=1}^{N} k_j \int_0^\infty \prod_{i=1}^{N}[F_i(t)]^{k_i}[1-F_i(t)]^{1-k_i}\frac{f_j(t)}{F_j(t)}dt. \tag{31}$$

Depending on the order distribution, population and sample sizes it is more comfortable to use one of the received alternative forms (27) or (31).
Formula (31) is the same as in Traat, Bondesson ja Meister (2000).

In the special case of all the distributions being equal, $F_i(t) = F(t),\ i = 1, 2, \ldots, N$, the formula (27) reduces to

$$p(\boldsymbol{k}) = (N-n)\int_0^\infty [F(t)]^n[1-F(t)]^{N-n-1}f(t)dt \tag{32}$$

and (31) to

$$p(\boldsymbol{k}) = n \int_0^\infty [F(t)]^{n-1} [1 - F(t)]^{N-n} f(t) dt. \tag{33}$$

Since $f(t) = \dfrac{dF(t)}{dt}$, we may write

$$p(\boldsymbol{k}) = (N - n) \int_0^1 [F(t)]^n [1 - F(t)]^{N-n-1} dF(t) \tag{34}$$

and

$$p(\boldsymbol{k}) = n \int_0^1 [F(t)]^{n-1} [1 - F(t)]^{N-n} dF(t). \tag{35}$$

Using relationship (Råde and Westergren 1999, p. 174)

$$\int_o^1 x^{m-1} (1 - x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{36}$$

we get from (34)

$$p(\boldsymbol{k}) = (N - n) \frac{\Gamma(n+1)\Gamma(N-n)}{\Gamma(N+1)} \tag{37}$$

and from (35)

$$p(\boldsymbol{k}) = n \frac{\Gamma(n)\Gamma(N-n+1)}{\Gamma(N+1)}.$$

Using $a\Gamma(a) = \Gamma(a+1) = a!$, if $a \in \boldsymbol{N}$, we get

$$p(\boldsymbol{k}) = \frac{\Gamma(n+1)\Gamma(N-n+1)}{\Gamma(N+1)} = \frac{n!(N-n)!}{N!} = \binom{N}{n}^{-1} \tag{38}$$

and

$$p(\boldsymbol{k}) = \frac{\Gamma(n+1)\Gamma(N-n+1)}{\Gamma(N+1)} = \binom{N}{n}^{-1}. \tag{39}$$

We see that the probability of getting a sample is exactly the same as for simple random sampling, i.e. in case of equal order distributions we get simple random sampling.

4.2 Uniform order sampling

Uniform order sampling was the first among order sampling scemes that was studied and used in practical survey. It was called sequential Poisson sampling and was not considered as a special case of the class of order sampling scemes (Ohlsson 1998).

With given λ, and $H(t)$ being uniform distribution, the terms in (31) are

$$F_j(t) = min(1, \lambda_j t),$$

$$1 - F_j(t) = max(0, 1 - \lambda_j t),$$

$$\frac{f_i(t)}{F_i(t)} = \begin{cases} \frac{1}{t}, \text{ if } t \in [0, \frac{1}{\lambda_i}], \\ 0, \text{ otherwise.} \end{cases}$$

Consequently, the probability function of the uniform order sampling design is

$$p(k) = \sum_{i=1}^{N} k_i \int_0^{\frac{1}{\lambda_i^*}} \frac{1}{t} \prod_{j=1}^{N} \min(1, \lambda_j t)^{k_j} \max(0, 1 - \lambda_j t)^{1-k_j} dt, \tag{40}$$

where $\frac{1}{\lambda_i^*} = \min(\frac{1}{\lambda_i}, \frac{1}{\lambda_j} : k_j = 0)$.

As soon as we know the sampling design it is possible to calculate the first and second order inclusion probabilities for population elements using (2)-(3).

4.3 The λ-estimator

Due to the absence of the analytical form of the probability function of order sampling design it was difficult to calculate the exact inclusion probabilities. Aires (1998) has given recursive algorithms only for Pareto order sampling. Therefore, it was generally not possible to use classical π-estimator under order sampling design.

Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the target inclusion probabilities. Since $\pi_i \approx \lambda_i$ and $\pi_i \to \lambda_i$ if the population size and sample size grow (Rosén 1998), it is reasonable to use λ-estimator for the population total Y:

$$\hat{Y}_\lambda = \sum_U \frac{1}{\lambda_i} y_i I_i. \tag{41}$$

The λ-estimator is used for the Consumer Price Index in Statistics Sweden together with uniform order sampling design (Ohlsson 1998).

As $EI_i = \pi_i$, $VarI_i = \pi_i(1-\pi_i)$ and $Cov(I_i, I_j) = \pi_{ij} - \pi_i\pi_j$, the design-based expectation and variance of $\hat{Y}_\lambda$ are

$$E\hat{Y}_\lambda = \sum_U \frac{\pi_i}{\lambda_i} y_i, \tag{42}$$

$$Var\hat{Y}_\lambda = \sum_U \frac{y_i y_j}{\lambda_i \lambda_j}(\pi_{ij} - \pi_i\pi_j). \tag{43}$$

The bias of λ-estimator is

$$E\hat{Y} - Y = \sum_{i=1}^{N} (\frac{\pi_i}{\lambda_i} - 1) y_i. \tag{44}$$

Ohlsson(1998) suggests the following formulae to compute the variance of λ-estimator:

$$Var\hat{Y}_\lambda = \frac{1}{n^2} \sum_{i=1}^{N} (\frac{n y_i}{\lambda_i} - Y)^2 \lambda_i (1-\lambda_i), \tag{45}$$

$$V\hat{a}r\hat{Y}_\lambda = \frac{1}{n(n-1)} \sum_{i=1}^{N} (\frac{n y_i}{\lambda_i} - \hat{Y})^2 (1-\lambda_i) I_i. \tag{46}$$

In the following examples we study the behaviour of λ-estimator under uniform order sampling design, and compare it with the behaviour of π-estimator. The necessary first and second order inclusion probabilities are calculated by (40) and (2)-(3).

5 Examples

In the following examples we study $OS\pi ps(n; H; \lambda)$ with $H(t)$ being uniform distribution. The corresponding sampling design is given by formula (40). In addition to the common πps target inclusion probabilities we consider mixed inclusion probabilities (10).

All the calculations were made in Mathcad (see Appendices).

5.1 Example 1

We consider population $U = \{1, 2, \cdots, 8\}$ and all possible samples with size $n = 3$, i.e. $\binom{8}{3} = 56$ samples all together.

Let $\mathbf{y}$ be the study variable related to the size variable $\mathbf{x}$ through the relationship

$$y_i = 2x_i + \varepsilon_i, \tag{47}$$

where $x_i \sim Exp(\frac{1}{5})$ and $\varepsilon_i \sim N(0,3)$.

In Table 1 one can see the finite population values $\mathbf{x}$ and $\mathbf{y}$ generated by (47) and target inclusion probabilities λ calculated by (8)-(9). The finite population total is 139.22 in our example.

Table 1. The values of $\mathbf{x}$, $\mathbf{y}$, λ and π

i	x_i	y_i	λ_i	π_i	$\frac{\lvert\lambda_i-\pi_i\rvert}{\lambda_i}$
1	33.35	63.85	1.0000	0.9070	0.093
2	8.22	11.38	0.4903	0.5410	0.103
3	2.68	5.49	0.1600	0.1501	0.062
4	5.24	10.13	0.3129	0.3119	0.003
5	0.97	3.62	0.0582	0.0528	0.093
6	8.74	24.05	0.5215	0.5823	0.117
7	1.71	5.84	0.1020	0.0938	0.080
8	5.95	14.86	0.3552	0.3611	0.016
$\sum$	66.87	139.22	3.0000	3.0000	

Using (40)and (2)-(3) the probabilities of all possible samples and the exact inclusion probabilities for population elements were found. Table 2 presents the symmetric matrix of second order inclusion probabilities; the main diagonal consists of the first order inclusion probabilities. One can compare given λ and realized π in Table 1 (see also Appendix 1).

Table2. Second order inclusion probabilities

	1	2	3	4	5	6	7	8
1	0.9070	0.4786	0.1236	0.2652	0.0426	0.5179	0.0763	0.3099
2	...	0.5410	0.0501	0.1111	0.0170	0.2627	0.0307	0.1318
3	...	...	0.1501	0.0270	0.0043	0.0560	0.0077	0.0316
4	...	...	...	0.3119	0.0092	0.1255	0.0166	0.0691
5	...	...	...	...	0.0528	0.0190	0.0027	0.0108
6	...	...	...	...	...	0.5823	0.0342	0.1494
7	...	...	...	...	...	...	0.0938	0.0195
8	...	...	...	...	...	...	...	0.3611

Every sample was used to compute estimates of population total and of the variances. Since in our case we know the exact inclusion probabilities then in addition to the λ-estimator we also consider the π-estimator. The design probabilities, the λ- and π-estimators (4) and (41) of population total and variance estimators (6) and (46) for some samples are printed in Table 3.

Table 3. Sampling design. The estimators

i	$\mathbf{k}$	$p(\mathbf{k})$	$\hat{Y}_\pi$	$\hat{Y}_\lambda$	$Va\hat{r}\hat{Y}_\pi$	$Va\hat{r}\hat{Y}_\lambda$
1	(1,1,1,0,0,0,0,0)	0.036	128.01	121.39	320.54	510.25
2	(1,1,0,1,0,0,0,0)	0.084	123.89	119.41	225.21	359.82
3	(1,1,0,0,1,0,0,0)	0.012	160.01	149.27	1594.68	491.28
4	(1,1,0,0,0,1,0,0)	0.223	132.73	133.18	160.33	217.87
5	(1,1,0,0,0,0,1,0)	0.022	153.72	144.37	1183.83	467.17
...	...	...	...	...	...	...
52	(0,0,0,1,0,0,1,1)	0.001	135.93	131.52	1056.79	536.36
53	(0,0,0,0,1,1,1,0)	0.000	172.19	165.66	758.28	544.29
54	(0,0,0,0,1,1,0,1)	0.001	151.06	150.18	1033.04	418.53
55	(0,0,0,0,1,0,1,1)	0.000	172.05	161.37	936.68	667.82
56	(0,0,0,0,0,1,1,1)	0.002	144.77	145.28	593.07	394.41

The main characteristics of the design-based distributions of the estimators are given in Table 4.

Table 4. The characteristics of the design-based distributions of the estimators

	$\hat{Y}_\pi$	$\hat{Y}_\lambda$	$Va\hat{r}\hat{Y}_\pi$	$Va\hat{r}\hat{Y}_\lambda$
Minimum	90.09	89.91	27.43	137.92
Expectation	139.22	136.24	284.51	295.72
Maximum	201.28	183.37	2681.86	836.66
Exact value	139.22	139.22	284.56	214.20

The exact values in Table 4 are the finite population total and the actual design based variances (5) and (43). The theoretical variance formula (45) gave the value $Var\hat{Y} = 225.91$.

Some notes about the example:

1. The exact inclusion probabilities differ from the target ones. The absolute relative differences between target and actual inclusion probabilities are less than 0.117 (see Table 1).

2. The λ-estimator of the population total has a bias, $B = -2.98$.

3. The λ-estimator of the population total has considerably smaller variance than the π-estimator (214.20 compared to 284.56).

4. $\hat{Var}\hat{Y}_\lambda$ (46) is seriously biased for actual design based variance (43), $B = 295.72 - 214.20 = 82.52$.

5. $\hat{Var}\hat{Y}_\lambda$ (46) estimates rather the design based variance (5) of the π-estimator than the variance (43) of λ-estimator.

6. $\hat{Var}\hat{Y}_\lambda$ is much more stable variance estimator than $\hat{Var}\hat{Y}_\pi$ (compare the minimums and maximums).

5.2 Example 2

In the second example the same population with variables **x** and **y** was considered. Mixed inclusion probabilities were used instead of the common πps inclusion probabilities.
Mixed inclusion probabilities were calculated by (10) where $\lambda_0 = \frac{3}{8}$, and λ_i were taken from Table 1. Mixing parameter μ was varied.
The λ-estimator, design-based variance (43), and variances (45)-(46) were studied. The results for different values of μ are given in Table 5.

Table 5. λ-estimator and its variance

	$\mu = 0$	$\mu = 0.01$	$\mu = 0.02$	$\mu = 0.04$	$\mu = 0.06$	$\mu = 0.1$
$E\hat{Y}_\lambda$ (42)	136.34	136.45	136.56	136.79	137.02	137.44
$Var\hat{Y}_\lambda$ (43)	214.49	208.90	204.49	201.14	202.58	224.27
$Var\hat{Y}_\lambda$ (45)	225.91	222.30	220.91	221.05	226.68	249.81
$E(\hat{Var}\hat{Y}_\lambda)$ (46)	295.72	283.52	275.53	269.80	274.90	309.09

The same interesting phenomena that the variance is not minimized with strictly πps probabilities, was also pointed out in Kröger, Särndal and Teikari (1999) for Pomix-sampling. In our case the design based variance (43) was minimized with $\mu \approx 0.04$.

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